

Relaxation of Electrons in the Boundary Layer of a Weakly Ionized Plasma

A. Schumacher

Institut für Theoretische Physik I, Ruhr-Universität Bochum

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We study the kinetics of fast electrons in a weakly ionized collision dominated plasma in front of a planar negative wall. For small Debye lengths the potential variation in the boundary layer can be approximated by a potential jump representing the space charge sheath. We propose an approximation method to calculate the electron distribution function which accounts for general boundary conditions (absorption, reflection and emission). This method is based on a parametrization of the angular dependency, the unknown parameters being determined by a suitable adaption of the Method of Weighted Residuals. The error inherent in our approximation procedure is discussed and shown to be small. As result we get a straightforward analytic representation of the electron distribution function in the boundary layer. For the special case of a totally absorbing wall our results are in good agreement with exact analytical solutions in the literature.

1. Introduction

Due to its wide range of applications (fusion research, probe theory, electrode and wall phenomena in gas discharges) the problem of the plasma boundary layer has received growing interest. The exact analytical description of the transition region between a plasma and a material wall presents a formidable problem, since the steep gradients and the strong space charge fields in front of the wall require a kinetic treatment which is further complicated by the need of a self-consistent potential solution.

Assuming a Boltzmann-distributed electron density the coupled problem of the *ion* kinetics and the self-consistent electric field has been solved in the boundary layer of a weakly ionized plasma for the collisionless [1] as well as for the collision dominated case [2]. A corresponding kinetic analysis of the *electrons*, however, which takes into account the strong inhomogeneous field, is still missing. The reason which makes the electron kinetics more complicated originates in the fact that due to the small mass ratio the electrons are essentially scattered elastically by the neutral particles. Therefore the collision mechanism requires a three dimensional treatment in velocity space, whereas under certain

conditions the ion kinetics can be investigated in a one-dimensional treatment [2].

Even if one simplifies the problem considerably by neglecting the electric field outside the space charge layer – which is justified for fast electrons – this difficulty still remains. In this case the electron Boltzmann equation can be treated analytically only by the rather involved Case-Zweifel formalism of singular integral equations or equivalent techniques known from neutron transport theory [3, 4]. Using these methods the electron distribution function in the boundary region has been calculated by Shcherbinin and Stakhanov [5, 6] in a model similar to ours (see Sect. II), where the electric field within the plasma is neglected and the potential variation in the sheath is replaced by a reflecting potential barrier which is taken into account by the boundary condition at the wall. As the result of their rather lengthy analytic calculations these authors get an integral equation which can only be solved numerically. Explicit results for the distribution function in the plasma or for the density variation are not given, these can be taken from a purely numerical treatment of Bakhst et al. [7]. Both papers are restricted to the special assumption of a completely absorbing wall, general boundary conditions are not discussed.

In view of the fact that previous investigations need a complicated and unhandy formalism and are restricted to a special boundary condition, it is the AIM OF OUR INVESTIGATION to develop a

Reprint requests to Dr. A. Schumacher, Institut für Theoretische Physik I, Ruhr-Universität Bochum, NB 7/156, D-4630 Bochum 1.

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straightforward approximation procedure for the calculation of the electron distribution function which allows to include general boundary conditions (reflection, absorption, emission). Moreover, since existing theories cannot be extended to account for the effect of a strong inhomogeneous electric field on the electron kinetics, special consideration will be given to the requirement that a corresponding generalization of our approximation method should eventually be possible. This generalization, however, will not be treated in this paper but will be the subject of a coming publication.

2. The Model

We consider a weakly ionized collision dominated plasma in front of a planar negative wall. In particular the following assumptions are made:

- The Debye length λ_D is small compared with all other lengths, consequently the potential variation in the space charge sheath is approximated by a potential jump $\Delta\Phi$ at the boundary.
- The variation of the potential in the plasma outside the sheath is small compared with the energy of the electrons and is neglected.
- The degree of ionisation is low so that collisions between electrons and neutrals are dominant. Due to the small mass ratio m/M (m, M denoting the mass of electrons and neutrals respectively) the energy exchange in these collisions is negligible. The collision cross section is taken here to be isotropic; our formalism, however, is not limited to this case, as can be seen from the development of our method in Section IV.

The above model specifications imply that the whole plasma boundary transition region can be split up into two different model zones: the “scattering zone” immediately in front of the sheath where the electrons are scattered by neutral particles without exchanging energy and the “energy exchange zone” where on a larger scale energy exchanging electron electron collisions come into play and make a transition to the plasma solution possible. In this paper we consider first the scattering zone.

Due to our model assumptions the energy of the electrons in the scattering zone is a free parameter which enters only into the boundary conditions, i.e. we consider a group of electrons with a specific kinetic energy $\varepsilon = \frac{1}{2} m v^2 \cong e \Delta\Phi$. Our solution then

has to describe how the strong anisotropy of the angular distribution induced by the boundary condition at the wall (e.g. half distribution for a completely absorbing wall) relaxes toward the plasma until finally far from the wall the nearly isotropic cosine distribution is reached. It should be noted that the so-called “Milne” and “Albedo” problem known from neutron transport theory [3,4] are recovered as special cases in our formalism.

3. Basic Equations

For the model specified in the previous section the Boltzmann equation for the electron distribution function f presents itself in the form [8]

$$\mu \frac{\partial f}{\partial x} = -f + \frac{1}{2} \int_{-1}^{+1} f d\mu. \quad (1)$$

Here we have introduced the dimensionless space coordinate $x = z/\lambda$, where $z (< 0)$ denotes the distance from the wall ($z = 0$) and $\lambda = 1/n q(\varepsilon)$ (n = neutral density, $q(\varepsilon)$ collision cross section) is the mean free path for momentum transfer; μ denotes the direction cosine of the angle between the electron velocity and the z -direction.

The boundary condition at the sheath edge reads

$$\begin{aligned} f^-(x=0, \mu) &= f^+(x=0, -\mu) & |\mu| &\leq \mu^*, \\ f^-(x=0, \mu) &= r f^+(x=0, -\mu) + \tilde{f}^w(\mu), & |\mu| &\geq \mu^*, \end{aligned} \quad (2)$$

where $f^\pm = f(\mu \gtrless 0)$, r is the coefficient of specular reflection at the wall*, and $\tilde{f}^w(\mu)$ is related to the angular distribution of emitted particles $f^w(\mu)$ by the equation

$$\tilde{f}^w(\mu) = f^w\left(\sqrt{\frac{\mu^2 - \mu^{*2}}{1 - \mu^{*2}}}\right) \quad (3)$$

which accounts for the free all in the sheath. The angle

$$\vartheta^* = \arccos \mu^*; \quad \mu^* = \sqrt{\frac{e \Delta\Phi}{\varepsilon}}, \quad (4)$$

which is determined by the ratio of the potential barrier $\Delta\Phi$ and the kinetic energy ε , separates the region in velocity space where particles are reflected in the sheath ($0 \leq \mu \leq \mu^*$) from the range ($\mu^* \leq \mu \leq 1$) where particles can reach the wall.

* Diffuse reflection is not considered here, since in general different energies are involved in the process.

Integrating equation (1) over the angular variable yields the equation of continuity with the solution

$$\Gamma = \int_{-1}^{+1} \mu f \, d\mu = \text{const.} \quad (5)$$

The constant Γ – the current of particles with a specific kinetic energy ε – is a free parameter since it is determined in the energy exchange zone which is not under consideration here.

Far away from the wall ($x \rightarrow -\infty$) the ansatz $f = f^0(x) + \delta f(x, \mu)$ with $\delta f \ll f^0$ yields with (5) the asymptotic form of the distribution function

$$f^{\text{as}}(x, \mu) = -\frac{3}{2}(x - \mu)\Gamma + \text{const} \quad (6)$$

which is an exact solution of (1) (and corresponds to the undamped eigenfunctions $f_1(x, \mu) = \frac{1}{2}$ and $f_2(x, \mu) = \frac{1}{2}(x - \mu)$ known from the Case-Zweifel formalism).

4. Construction of an Approximation Procedure

a) The Method of Weighted Residuals

Our approximation method for the calculation of the electron distribution in the boundary layer is based on a parametrization of the μ -dependency. The unknown coefficients introduced by this procedure are determined from the requirement that the deviations from the exact solution are as small as possible. In the modern literature [9, 10] such methods have been summarized under the name “Method of Weighted Residuals” (MWR).

In the following we will give a short sketch of the Method of Weighted Residuals as far as we need it for our purposes. More details can be found in the literature.

We seek a solution of the Boltzmann equation (1) which can be written in the symbolic form

$$\mathcal{B}(f) = 0 \quad (7)$$

with the appropriate boundary conditions given by (2). We introduce an ansatz f^* for the distribution function f which accounts for the μ -dependency in a parametrized form

$$f^*(x, \mu) = \sum_{j=1}^k c_j(x) \psi_j(\mu). \quad (8)$$

Here the ψ_j are given functions (so called “trial functions”), the coefficients or “parameter functions” c_j have still to be determined. In contrast to the

exact solution f the ansatz f^* does not fulfil (1) identically, but rather yields

$$\mathcal{B}(f^*) = \mathcal{R}(x, \mu), \quad (9)$$

where \mathcal{R} is called the residual. The idea of the MWR now requires that this residual which would be identically zero for the exact solution becomes zero in some average sense. For k given weighting functions W_j one must have

$$\int_{-1}^{+1} \mathcal{R}(x, \mu) W_j(\mu) \, d\mu = 0, \quad j = 1, \dots, k. \quad (10)$$

With these k equations the k unknown functions c_j can be determined.

From this short review one can already see that the Method of Weighted Residuals does not give an explicit prescription for the construction of an approximate solution, but only a rough outline of general aspects. In fact, such different methods as the expansion into orthogonal eigenfunctions and the so-called “two-Maxwellian approaches” known e.g. from probe theory can be looked upon as special cases of the MWR.

Therefore applying the MWR to a specific problem one must exploit the freedom left by the method to adapt the ansatz to the physics of the problem. Moreover, one should be able to argue the choice of the weighting functions, to estimate their influence on the results, and to discuss the error introduced by the approximation procedure.

b) The Ansatz

For our problem we try to find an ansatz which meets the following requirements:

1. It must be physically motivated, that means it has to account for the most important physical aspects of the boundary region.
2. For $x \rightarrow -\infty$ it must approach the asymptotic solution (6).
3. The boundary conditions (2) have to be fulfilled.
4. The ansatz should yield a form of the residual (compare (9)) which makes an estimate of this quantity possible.
5. The choice of the weighting function and its influence on the result should be transparent.
6. The ansatz should consist of two unknown parameter functions.

The first three requirements are easily understood, the points 4 and 5 are necessary to estimate the error of our approximation procedure. The condition 6 is not crucial for the investigation presented in this paper but since our final aim is a generalization of the method which includes the electric field this requirement becomes important. In that case the equations to determine the parameter functions of the ansatz lead to a system of coupled partial differential equations which must remain analytically tractable.

For the construction of our ansatz we start from the asymptotic angular dependency of the distribution function (6) and account for the influence of the wall by a "disturbance" δ whose structure we will make transparent in the following. For that purpose we consider first the case $\mu^* \equiv 0$ (no potential barrier) and a totally absorbing wall. Due to the absorption there are only few particles near the wall in the half space $\mu \leq 0$ since all trajectories which come from the wall are empty and must be refilled by collisions. Formally we can treat the wall as a source of "missing" electrons with negative density contribution which are "destroyed" by collisions. The probability that a particle covers the distance from the wall to a point x without a collision is given by $\exp\{-x/\mu\}$. Taking into account the boundary condition $f^-(0) = 0$ we get the negative contribution to the distribution function due to these missing particles in the form $\delta = -(F^{00} + \mu F^{10}) \exp\{-x/\mu\}$ for $\mu < 0$ where F^{00} and F^{10} are the boundary values of F^0 and F^1 for $x = 0$. If the wall is not totally absorbing but reflects and emits particles the corresponding generalization yields

$$\delta = \{(r-1) F^{00} - (r+1) \mu F^{10} + \tilde{f}^w(\mu)\} \exp\{-x/\mu\}.$$

Let us now consider a reflecting potential barrier $\mu^* \neq 0$. In this case the above argumentation does not hold for the whole negative μ -half space, but only for the region $-1 \leq \mu \leq -\mu^*$. In the region $-\mu^* \leq \mu \leq 0$ which corresponds to particle trajectories which return within the sheath the boundary condition $f^- = f^+$ requires the symmetry of the distribution function around $\mu = 0$ and thus leads to a boundary value of $\delta(0) = -2\mu F^{10}$. The relaxation of this wall disturbance by collisions ($\sim \exp\{-x/\mu\}$) then yields $\delta = -2\mu F^{10} \exp\{-x/\mu\}$.

Of course, one should bear in mind that due to the conservation of particles electrons and "missing

electrons" which after a collision are scattered out of the part δ of the distribution function have to be accounted for in an integral form by the quantities F^0 and F^1 .

We are thus led to the following ansatz

$$f^* = F^0(x) + \mu F^1(x) + \delta(x, \mu) \quad (11a)$$

with

$$\delta = \begin{cases} 0, & \mu > 0, \\ -2\mu F^{10} \exp\{-x/\mu\}, & -\mu^* \leq \mu \leq 0, \\ [(r-1)F^{00} - (r+1)\mu F^{10} + \tilde{f}^w(\mu)] \exp\{-x/\mu\}, & -1 \leq \mu \leq -\mu^*, \end{cases} \quad (11b)$$

where

$$F^{00} = F^0(0), \quad F^{10} = F^1(0). \quad (11c)$$

From a more formal point of view the underlying physical picture of our ansatz may also be characterized as follows: The function δ is a solution of the equation

$$\mu (\partial \delta / \partial x) + \delta = 0. \quad (12)$$

Since the collision term of this equation ($-\delta$) accounts for losses only, wall disturbances relax with the characteristic length x/μ .

c) Determination of the Parameter Functions

Due to the construction our ansatz fulfils automatically the requirements 1, 2, 3 and 6 of the previous section. That the same holds also for conditions 4 and 5 will become clear in the following. We determine the parameter function F^0 and F^1 such that the residual which is defined by

$$\mathcal{R}(x, \mu) = \mu \frac{\partial f^*}{\partial x} + f^* - \frac{1}{2} \int_{-1}^{+1} f^* d\mu \quad (13)$$

is small compared with a typical term of the r.h.s. of (13). Then the argument is the following: Our ansatz f^* is an exact solution of (13) which differs from the equation for f (1) only by the negligibly small term $\mathcal{R}(x, \mu)$, moreover f and f^* fulfil the same boundary conditions. Thus the ratio of the residual to a typical term of the kinetic equation is an estimate for the error inherent in our approximation procedure. For this estimate of the error the fulfilment of the boundary condition is important since a small residual alone does not guarantee a good approximation of the distribution function as one can see from the example $f = f^{\text{as}}$ (see (6)). Although the residual is identically zero this ap-

proximation is extremely bad for the boundary region. The reason is of course that the boundary conditions are violated.

Inserting our ansatz (11) into (13) we get for the residual a polynomial of second order in μ . For the following, however, it is more convenient to write it in the equivalent form

$$\begin{aligned} \mathcal{R}(x, \mu) = & \alpha_0(x) P_0(\mu) + \alpha_1(x) P_1(\mu) \\ & + \alpha_2(x) P_2(\mu). \end{aligned} \quad (14)$$

The coefficients of the Legendre Polynomials $P_n(\mu)$ are given by

$$\begin{aligned} \alpha_0(x) &= \frac{1}{3} \frac{\partial F^1}{\partial x} + F^0 - \frac{1}{2} \int_{-1}^{+1} f^* d\mu \\ &= \frac{1}{3} \frac{\partial F^1}{\partial x} - \frac{1}{2} \int_{-1}^{+1} \delta d\mu, \\ \alpha_1(x) &= \frac{\partial F^0}{\partial x} + F^1, \\ \alpha_2(x) &= \frac{2}{3} \frac{\partial F^1}{\partial x}. \end{aligned} \quad (15)$$

To determine F^0 and F^1 we need two weighting functions $W_1(\mu)$, $W_2(\mu)$ which according to (10) yield the missing equations for the parameter functions. One of the weighting functions is determined by the physics of the problem which requires the conservation of the number of particles (of a given kinetic energy ε) and therewith the constancy of the corresponding current contribution Γ .

Setting $W_1 = 1$ we get from (10) the continuity equation

$$\int_{-1}^1 \mathcal{R}(x, \mu) d\mu = \int_{-1}^1 \mu \frac{\partial f^*}{\partial x} d\mu = 0, \quad (16)$$

$$\Rightarrow \Gamma = \int_{-1}^1 \mu f^* d\mu = \text{const}. \quad (17)$$

Due to the orthogonality of the Legendre polynomials this yields as equation for our parameter function $F^1(x)$

$$\alpha_0(x) = \frac{1}{3} \frac{\partial F^1}{\partial x} - \frac{1}{2} \int_{-1}^{+1} \delta d\mu = 0. \quad (18)$$

Inserting (11 b) and integrating we find

$$\begin{aligned} F_1(x) &= \frac{3}{2} \Gamma + \frac{3}{2} F^{00} (r-1) \\ &\quad \cdot \{E_3(-x) - \mu^{*2} E_3(-x/\mu^*)\} \\ &\quad + \frac{3}{2} F^{10} \{(r+1) E_4(-x) + (1-r) \mu^{*3} E_4(-x/\mu^*)\} \\ &\quad - \frac{3}{2} \int_{-1}^{-\mu^*} \mu \tilde{f}^w(\mu) e^{-x/\mu} d\mu, \end{aligned} \quad (19)$$

where the functions $E_n(x)$ are given by [11]

$$E_n(x) = \int_0^1 \mu^{n-2} e^{-x/\mu} d\mu. \quad (20)$$

For the choice of the second weighting function W_2 a corresponding physical argumentation is not available. Therefore we do not specify it at the moment but expand it into Legendre polynomials:

$$W_2(\mu) = \sum_{n=0}^{\infty} w_n P_n(\mu). \quad (21)$$

The averaging procedure (10) then yields as the second equation for our parameter functions

$$\alpha_1 + \frac{3}{5} \frac{w_2}{w_1} \alpha_2 = \frac{\partial F^0}{\partial x} + F^1 + \frac{2}{5} \frac{w_2}{w_1} \frac{\partial F^1}{\partial x} = 0, \quad (22)$$

where only the ratio w_2/w_1 of two coefficients of the Legendre polynomial expansion enters. Let us now consider the residual (14). Taking into account the Eqs. (18), (22) we get

$$\mathcal{R}(x, \mu) = R(x) \left\{ P_2(\mu) - \frac{3}{5} \frac{w_2}{w_1} P_1(\mu) \right\} \quad (23)$$

with

$$R(x) = \frac{2}{3} \frac{\partial F^1}{\partial x}, \quad (24)$$

where the factor $R(x)$ does not depend – neither explicitly nor implicitly – on w_2/w_1 as one can see by inserting (19). This means, the choice of the weighting function has an influence on the magnitude of the residual only through the μ -depending part. Since we have to minimize the residual we require

$$\int_{-1}^{+1} \mathcal{R}^2 d\mu = \min. \quad (25)$$

Due to the orthogonality of the $P_n(\mu)$ this yields

$$w_2/w_1 = 0 \quad (26)$$

(26) does not determine an explicit weighting function. All weighting functions whose coefficient w_2 disappears are equivalent for our problem and do not influence the results at all. All other weighting functions increase the residual which represents the error of our procedure.

With (26) the second equation (22) for our parameter functions F^0, F^1 reduces to

$$\partial F^0 / \partial x + F^1 = 0. \quad (27)$$

Inserting F^1 from (19) we find

$$\begin{aligned} F^0 = & -\frac{3}{2} \Gamma x + c_1 - \frac{3}{2} F^{00} (r-1) \\ & \cdot \{E_4(-x) - \mu^{*3} E_4(-x/\mu^*)\} \\ & - \frac{3}{2} F^{10} \{(r+1) E_5(-x) + (1-r) \mu^{*4} E_5(-x/\mu^*)\} \\ & - \frac{3}{2} \int_{-1}^{-\mu^*} \tilde{f}^w(\mu) \mu^2 e^{-x/\mu} d\mu. \end{aligned} \quad (28)$$

The solution is not yet completely determined since we have to know the constant c_1 in (28) and the boundary values F^{00} and F^{10} . For these three unknowns, however, we have only the two Eqs. (11c).

Using these equations to eliminate two quantities we still can choose e.g. F^{10} arbitrarily without violating the fulfilment of the boundary conditions or the approach to the correct asymptotic solution.

Trivially not any choice of F^{10} can lead to a good approximation of the distribution function. The criterion for a good or bad choice of course is the magnitude of the residual. We have already minimized R with respect to the μ -dependency, the freedom in the choice of F^{10} must be used to assure the smallness of the residual for all x . Consequently we determine F^{10} through the requirement

$$\int_{-\infty}^0 R^2 dx = \min. \quad (29)$$

Details of this evaluation are given in the appendix where we also show that for all x the residual of our solution for f is small compared with a typical term of the kinetic equation, which demonstrates the validity of our approximation procedure.

5. Results and Discussion

With (19) and (28) and the knowledge of the three constants c_1, F^{00} , and F^{10} (see Appendix) our ansatz (11) is completely determined, which means

that the spatial and angular dependency of the electron distribution function in the scattering zone is known.

Before we present our results for various boundary conditions, let us first consider the special case of a completely absorbing wall ($r=0, f^w=0, \Gamma=1$; “Milne problem”), where we can compare with an exact solution of (1). Figure 1 shows the exact distribution function calculated numerically by Bakhst et al. [7] and our approximate solution for the value $\mu^*=0.5$ and for various distances from the wall. We find an excellent agreement of the curves which in particular accounts for the most characteristic aspects: The filling of the range $-1 \leq \mu \leq -\mu^*$ the decrease of the symmetry in the range $-\mu^* \leq \mu \leq \mu^*$ and the transition to the cosine distribution after few mean free paths.

Figure 2 shows our results for essentially the same case; however, we used polar coordinates which seems to be more instructive, and included in addition specular reflection at the wall with a reflection coefficient $r=0.2$. In this representation the

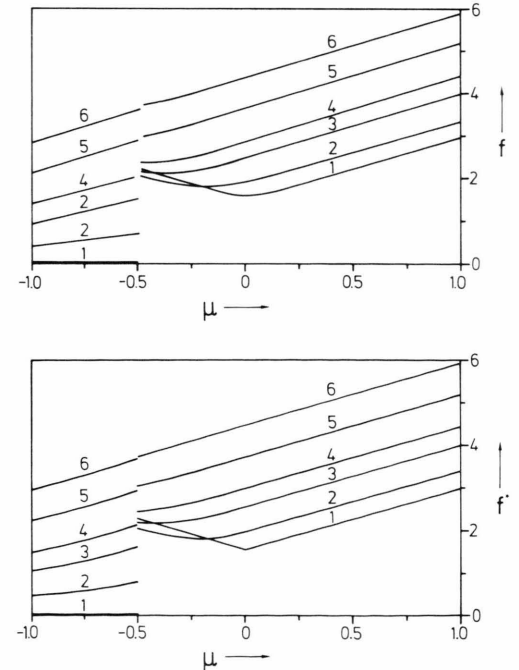


Fig. 1. Comparison of our approximate solution (f^*) with the exact distribution function (f) [7] for various distances from the wall:

1: $x=0$ 3: $x=-0.7$ 5: $x=-1.5$
 2: $x=-0.3$ 4: $x=-1$ 6: $x=-2$
 The other parameters are: $\Gamma=1, r=0, \tilde{f}^w=0, \mu^*=0.5$.

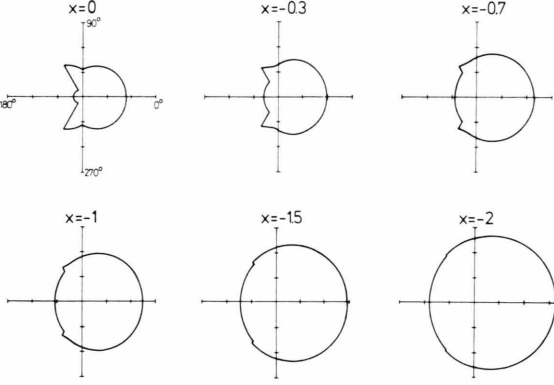


Fig. 2. Polar coordinate representation of the distribution function. The parameters are the same as in Fig. 1 with the exception $r = 0.2$.

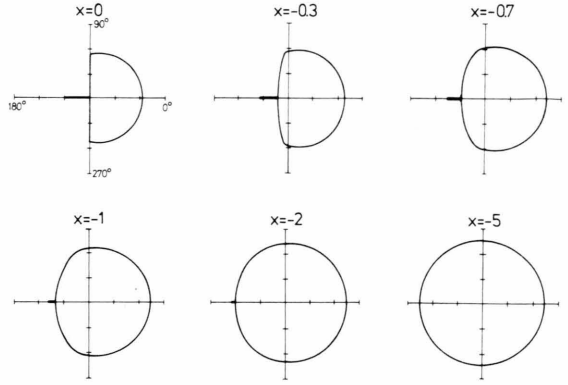


Fig. 3. Distribution function for the "albedo problem" ($\Gamma = 0$, $\tilde{f}^w = \delta(\mu + 1)$, $r = 0$, $\mu^* = 0$). The fat line at $\beta = \arccos(\mu) = 180^\circ$ characterizes the contribution of the beam particles.

influence of the wall is reflected in the cone-like structure of the distribution function. With increasing distance from the wall this cone is filled by collisions until the nearly isotropic cosine distribution is recovered.

As another representative example of our method we consider the wellknown "Albedo-problem" [3], where a beam of particles injected from the wall at a definite angle μ_0 is scattered in the plasma and diffuses back to the wall (i.e. $f^w(\mu) = F^b \delta(\mu - \mu_0)$, $r = 0$, $\Gamma = 0$). Figure 3 shows the angular relaxation of the distribution function for $F^b = 1$, $\mu^* = 0$ and $\mu_0 = -1$ where the fat line at $\mu = -1$ characterizes the remaining strength of the beam. Figure 4 shows the corresponding density* variation with the injection angle μ_0 as a parameter. The maximum near or at the wall for angles $|\mu_0| \lesssim 0.7$ is explained by the fact that the density contribution of the beam particles, which falls off towards the plasma dominates the always negative density gradient of the counter diffusing plasma particles.

Let us summarize: Starting from a physically motivated ansatz which fulfils the boundary conditions and approaches the correct asymptotic solution we construct an approximation procedure to calculate the electron distribution function. This method does not only allow to discuss how the choice of the weighting functions within the frame of the MWR influences the results but also to estimate the error inherent in our approximation scheme. The requirement to minimize this error

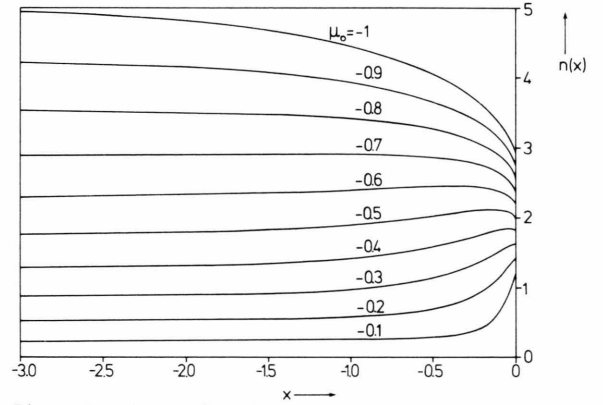


Fig. 4. Density profiles for the albedo problem with the injection angle as parameter ($\Gamma = 0$, $\tilde{f}^w = \delta(\mu - \mu_0)$, $r = 0$, $\mu^* = 0$).

leads to the equations which determine the parameter functions. As result we get with (19) and (28) a straightforward analytic representation of the distribution function which can account for absorption, reflection and emission at the wall. For the special case of an absorbing wall our results are in good agreement with exact calculations.

One point more in comment: It should have become clear from the development of our method that in contrast to previous theories a generalization which includes the effect of an electric field on the electron kinetics is quite possible, since the argumentation which led to the construction of our ansatz is not necessarily restricted to the field free case. This generalized treatment of the scattering zone together with the simultaneous analysis of the

* Note that "density" in our model always means the density contribution of particles with a specific energy ε .

energy exchange zone is currently in progress. Both extensions of the theory are necessary for a consistent analysis, which yields the electron distribution function in the whole boundary layer from the nearly undisturbed plasma body to the wall.

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Appendix

With the Eqs. (19), (28) we have found our parameter functions F^0 and F^1 . It still remains to determine the constants F^{00} , F^{10} and c_1 . A relation between these constants is given by the equations

$$\begin{aligned} F^0(0) = F^{00} &= c_1 + \frac{1}{2} F^{00} (1-r) (1-\mu^{*3}) \\ &\quad - \frac{3}{8} F^{10} \{r+1 + (1-r) \mu^{*4}\} - \frac{3}{2} F_2^w(0), \\ F^1(0) = F^{10} &= \frac{3}{2} \Gamma - \frac{3}{4} F^{00} (1-r) (1-\mu^{*2}) \\ &\quad + \frac{1}{2} F^{10} \{r+1 + (1-r) \mu^{*3}\} - \frac{3}{2} F_1^w(0), \end{aligned} \quad (\text{A1})$$

where we introduced the abbreviation

$$F_n^w(x) = \int_{-1}^{-\mu^*} \mu^n \tilde{f}^w(\mu) \exp(-x/\mu) d\mu. \quad (\text{A2})$$

With (A1) we can express F^{00} and c_1 by F^{10} and get

$$\begin{aligned} F^{00} &= \frac{2\Gamma}{(1-r)(1-\mu^{*2})} - \frac{2}{3} F^{10} \frac{1-\mu^{*3}}{1-\mu^{*2}} \\ &\quad - \frac{2}{(1-r)(1-\mu^{*2})} F_1^w(0), \end{aligned} \quad (\text{A3a})$$

$$\begin{aligned} c_1 &= F^{00} \left\{1 - \frac{1}{2} (1-r) (1-\mu^{*3})\right\} \\ &\quad + \frac{3}{2} F^{10} \{r+1 + (1-r) \mu^{*4}\} + \frac{3}{2} F_2^w(0), \end{aligned} \quad (\text{A3b})$$

where F^{00} has to be inserted from (A3a).

For the x -depending part $R(x)$ of the residual $\mathcal{H}(x, \mu) = R(x) P_2(\mu)$ we get the explicit form

$$R(x) = \frac{2}{3} \frac{\partial F^1}{\partial x} = R_0 + R_1 F^{10} \quad (\text{A4})$$

with

$$\begin{aligned} R_0 &= \left\{ -\frac{2\Gamma}{(1-\mu^{*2})} + \frac{2}{1-\mu^{*2}} F_1^w(0) \right\} \\ &\quad \cdot [E_2(-x) - \mu^* E_2(-x/\mu^*)] + F_0^w(x), \\ R_1 &= \frac{2}{3} (1-r) \frac{1-\mu^{*3}}{1-\mu^{*2}} \{E_2(-x) - \mu^* E_2(-x/\mu^*)\} \\ &\quad + (r+1) E_3(-x) + (1-r) \mu^{*2} E_3(-x/\mu^*). \end{aligned} \quad (\text{A5})$$

The requirement

$$\int_{-\infty}^0 \mathcal{H}^2 dx = P_2(\mu) \int_{-\infty}^0 R^2 dx = \min \quad (\text{A6})$$

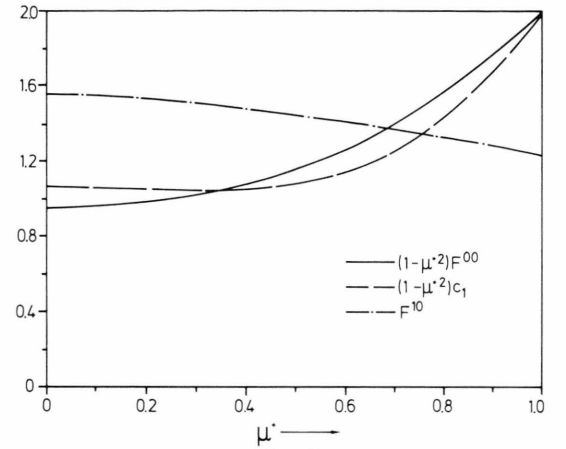


Fig. A1. The constants F^{00} , F^{10} and c_1 as functions of μ^* ($\Gamma=1$, $r=0$, $\tilde{f}^w=0$).

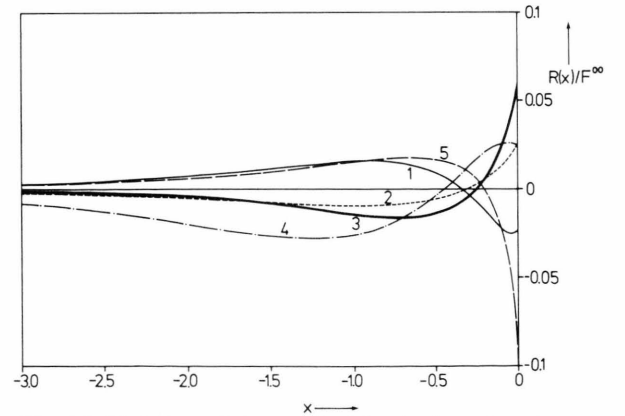


Fig. A2. Variation of the residual in the boundary layer. The curves correspond to the following cases:

- 1: $\Gamma=1$, $r=0$, $\tilde{f}^w=0$, $\mu^*=0.2$,
- 2: $\Gamma=1$, $r=0$, $\tilde{f}^w=0$, $\mu^*=0.9$,
- 3: $\Gamma=1$, $r=0.2$, $\tilde{f}^w=0$, $\mu^*=0.5$,
- 4: $\Gamma=0$, $r=0$, $\tilde{f}^w=\delta(\mu+1)$, $\mu^*=0.2$,
- 5: $\Gamma=1$, $r=0$, $\tilde{f}^w=1$, $\mu^*=0$.

yields the following equation for F^{10}

$$F^{10} = - \int_{-\infty}^0 R_0 R_1 dx \bigg/ \int_{-\infty}^0 R_1^2 dx. \quad (\text{A } 7)$$

As an example the values of F^{10} for a special case are shown together with the corresponding quantities F^{00} and c_1 in Figure A.1.

To discuss the error of our procedure we have to compare the residual with a typical term of the

kinetic equation. In Fig. A.2 the x -depending part $R(x)$ of the residual is shown for various boundary conditions and various values of μ^* . As a typical term of the kinetic equation we chose F^{00} which gives an upper limit for the error of our approximation procedure, since in general $F^{00} \leq F^0(x)$ holds. One can see that the residual is small in the whole region in front of the wall, which means that our approximation method is valid.

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